Mixing in Turbulent Axially Symmetric Free Jets

GDALIA KLEINSTEIN*
Polytechnic Institute of Brooklyn, Freeport, N.Y.

An analysis to solve mixing problems involving momentum, energy, and mass transfer in turbulent axially symmetric compressible flow is presented. The method is based on the linearization of the conservation equations in the plane of the von Mises variables while retaining the essential nonlinear nature of the equations in the physical plane. The application of this method to a laminar flow problem has been shown to be in good agreement with a numerical solution of the boundary-layer equations obtained by Pai. The extension of the method to turbulent flow requires an expression for the eddy viscosity. A possible modification of Prandtl's formulation, suggested by Ferri, is investigated. When this expression for eddy viscosity is introduced, the application of the present method to compressible turbulent flow yields analytical results that are in good agreement with available experimental data.

Nomenclature

A_k = either one of quantities 1, Pr^{-1}, Sc^{-1}
c_p = specific heat at constant pressure
h_i = static enthalpy of i species
H = stagnation enthalpy
I_0 = modified Bessel function of first kind
L_e = turbulent Lewis number $(\rho\epsilon_y)/(\rho\epsilon_h)$
P = defined by Eq. (10a)
P_k = either one of variables u, H, or Y_i
P_0 = defined by Eq. (10b)
Pr = turbulent Prandtl number $(\rho\epsilon)/(\rho\epsilon_h)$
P^* = P/P_0
r = radial coordinate
$\bar{r}$ = nondimensionalized radial coordinate $\bar{r} = r/r_j$
Sc = turbulent Schmidt number $(\rho\epsilon)/(\rho\epsilon_y)$
T = static temperature
$\bar{T}$ = nondimensionalized static temperature $\bar{T} = T/T_j$
u = axial velocity component
$\bar{u}$ = nondimensionalized axial velocity component $\bar{u} = u/u_j$
v = radial velocity component
$\bar{v}$ = nondimensionalized radial velocity component $\bar{v} = v/u_j$
w = molecular weight of mixture $\Sigma(Y_i/w_i)^{-1}$
w_i = molecular weight of i species
$\bar{w}$ = nondimensionalized molecular weight $\bar{w} = w/w_j$
x = axial coordinate
$\bar{x}$ = nondimensionalized axial coordinant $\bar{x} = x/r_j$
Y = mass concentration of jet's fluid
Y_i = mass concentration of i species
ξ_k = transformed axial coordinate defined by Eq. (7)
ρ = density
$\bar{\rho}$ = nondimensionalized density $\bar{\rho} = \rho/\rho_j$
$\overline{\rho\epsilon}$ = turbulent transfer coefficient for momentum $\rho\epsilon/\rho_j u_j r_j$
$\overline{\rho\epsilon_h}$ = turbulent transfer coefficient for energy $\rho\epsilon_h/\rho_j u_j r_j$
$\overline{\rho\epsilon_y}$ = turbulent transfer coefficient for concentration $\rho\epsilon_y/\rho_j u_j r_j$
ψ = stream function
ψ' = dummy variable
Ψ = stream function for axially symmetric flow defined by $\psi = (\Psi/2)^2$
ϵ = coefficient of kinematic viscosity of turbulent flow

Subscripts

0 = properties on axis
$\frac{1}{2}$ = properties at half-width of profile
e = external conditions
h = corresponding to enthalpy
j = initial properties of jet
v = corresponding to velocity
y = corresponding to mass concentration

Introduction

In many boundary-layer problems the assumption of flow similarity cannot be applied throughout the flow field; the validity of the similar solution is restricted to a limited region of the flow field, and the location of this region is not known a priori. Solutions for the nonsimilar cases can be obtained either by a laborious finite difference technique or by approximate methods. One of the useful techniques for deriving solutions for nonsimilar problems is the integral method; however, this application is limited in practice to reasonably smooth profiles.[1, 2] Therefore, problems involving the determination of the concentration decay in the mixing of two uniform streams, where the initial profile is physically a step function, lie outside the scope of the integral method.

A different approach[3-5] consists of linearizing the momentum equation in its differential form and obtaining the exact solution for given (arbitrary) initial conditions. According to the "modified Oseen method" of Carrier,[5] the convective operator in the original partial differential equation is replaced by a linear one. The resulting heat-conduction equations can be treated by well-known techniques.

For axially symmetric laminar compressible jets, Lewis and Carrier's technique was modified[2] by applying the von Mises transformation[6] to the exact differential equations with a subsequent linearization of the viscous term. The resulting equations in the transformed plane are linear and explicitly independent of the thermodynamic properties of the fluid, but the nonlinear nature of the equations is retained in the physical plane. A comparison with a finite difference solution due to Pai[7] indicates that, at least for the case mentioned, the method is sufficiently accurate.

If turbulent transfer coefficients are defined in a manner analogous to the laminar case, the equations become formally identical to those for laminar flow. Determination of the necessary transfer coefficients by a theoretical analysis of the properties of turbulence is a formidable task; therefore, a semiempirical approach is usually employed.

Libby[8] analyzed the compressible turbulent mixing for reactive gases. A similar approximation, but a different method than in Ref. 2, was used in order to linearize the equations in the von Mises variables. The eddy viscosity

Received July 10, 1963; revision received April 8, 1964. This research was sponsored by the Aeronautical Research Laboratories, Office of Aerospace Research, U.S. Air Force, under Contract AF 33(616)-7661, Project 7064, and is partially supported by the Ballistic Systems Division; also, it was taken from the dissertation submitted to the Faculty, Polytechnic Institute of Brooklyn, in partial fulfillment of the requirements for Doctor of Philosophy (Astronautics), 1963. The author wishes to express his gratitude to Antonio Ferri for his patience and careful guidance throughout his supervision of this paper. He also wishes to thank Lu Ting for his many suggestions and encouragement and Paul A. Libby for his valuable discussions.

* Research Associate. Presently Research Assistant Professor at New York University. Member AIAA.

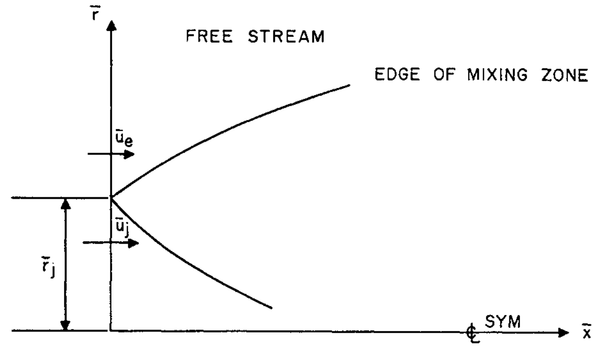

Fig. 1 Schematic diagram of the flow showing the coordinate system.

was obtained through the application of a compressibility transformation to the kinematic eddy viscosity, which in turn was empirically established from incompressible flow data.

A different approach, suggested by Ferri, considers a formulation of the eddy viscosity in terms of compressible flow variables and an arbitrary constant which must be determined experimentally. The approach is considered more direct than Ref. 8, because it does not rely upon an extrapolation of empirical data from incompressible to compressible flow. The present paper extends the analysis of Ref. 2 to turbulent flow, investigates a possible extension of Prandtl's incompressible formulation[9] of the turbulent viscosity coefficient to compressible flow, and determines the numerical values of some necessary constants appearing in the formulation of such coefficients. Numerous examples based on these coefficients are then presented and compared with available experimental results.

The examples considered in this paper are restricted to problems of injection into a quiescent media. The method, however, can also be used to obtain a solution for cases involving a moving external flow, provided that the eddy viscosity used can be expressed as a function of the axial coordinate only.

Analysis

The problem of an axisymmetric jet in a compressible, inhomogeneous, nonreacting turbulent fluid can be described by the boundary-layer equations (a schematic representation is shown in Fig. 1). If the turbulent Prandtl, Schmidt, and Lewis numbers are not much different from unity and the axial pressure gradient is neglected,† these equations may be written as

$$\partial(\rho u r)/\partial x + \partial(\rho v r)\partial r = 0 \quad (1a)$$

$$\rho u \partial u/\partial x + \rho v \partial u/\partial r = r^{-1}\partial[(\rho\epsilon)r^{\partial u}/\partial r]\partial r \quad (1b)$$

$$\rho u \partial H/\partial x + \rho v \partial H/\partial r = r^{-1}\partial[Pr^{-1}(\rho\epsilon)r\partial H/\partial r]/\partial r \quad (1c)$$

$$\rho u \partial Y_i/\partial x + \rho v \partial Y_i/\partial r = r^{-1}\partial[Sc^{-1}(\rho\epsilon)r\partial Y_i/\partial r]\partial r \quad (1d)$$

Equations (1b), (1c), and (1d) are formally identical and have the general form

$$\rho u \partial P_k/\partial x + \rho v \partial P_k/\partial r = r^{-1}\partial[A_k(\rho\epsilon)r\partial P_k/\partial r]\partial r \quad (2)$$

where P_k is either u, H, or Y_i, and A_k is the corresponding coefficient appearing in Eqs. (1b), (1c), and (1d). The transformation and linearization techniques of Ref. 2 are applied here to Eq. (2). A dimensionless stream function $\psi = (\Psi/2)^2$ is introduced to obtain the relation

$$\partial(\Psi/2)^2/\partial r = \bar{\rho}\bar{u}\bar{r} \qquad \partial(\Psi/2)^2/\partial\bar{x} = -\bar{\rho}\bar{v}\bar{r} \quad (3)$$

† This neglect of $\partial P/\partial x$ for an open jet permits the analysis that follows, and the subsequent agreement between the theory and the experimental data (Figs. 4, 5, and 6) appear to justify it, as does other literature.[10]

Applying the von Mises transformation, with Ψ and $\bar{x}$ as independent variables, Eq. (2) becomes

$$\partial P_k/\partial\bar{x} = 4\Psi^{-1}\partial[A_k\overline{\rho\epsilon}\,\bar{\rho}\bar{u}\bar{r}^2\Psi^{-1}\partial P_k/\partial\Psi]/\partial\Psi \quad (4)$$

where

$$\bar{r}^2 = \int_0^\Psi \frac{\Psi d\Psi}{\bar{\rho}\bar{u}}$$

A Taylor series expansion in Ψ of the term $A_k(\overline{\rho\epsilon})\bar{\rho}\bar{u}\bar{r}^2$ in the neighborhood of a point $\bar{x}$ on the axis yields

$$[A_k(\overline{\rho\epsilon})\bar{\rho}\bar{u}\bar{r}^2]_{x,\psi} = [A_k(\overline{\rho\epsilon})]_{x,0}\,\Psi^2/2 + 0(\Psi^4) \quad (5)$$

When terms of the fourth order in Ψ are neglected, Eq. (4) reduces to the linear equation[2, 11, 12]:

$$\partial P_k/\partial\xi_k = \Psi^{-1}\partial[\Psi\partial P_k/\partial\Psi]/\partial\Psi \quad (6)$$

where

$$\xi_k = 2\int_0^{\bar{x}} (A_k\overline{\rho\epsilon})_{\xi,0}d\bar{x} \quad (7)$$

which is in the form of the well-known heat conduction equation in cylindrical coordinates. Solutions for Eq. (6) are available.[13, 14] Subject to the boundary conditions, $P_k(\psi, 0) = g_k(\psi)$ and $P_k(\infty, \xi_k) = P_{ke}$, and the regularity condition on the axis, $P_k(0, \xi_k) \neq \infty$, it has the general solution

$$P_k(\Psi, \xi_k) = \frac{1}{2\xi_k}\exp\left(-\frac{\Psi^2}{4\xi_k}\right)\int_0^\infty \exp\left(-\frac{\psi'^2}{4\xi_k}\right)I_0 \times \left(\frac{\psi\psi'}{2\xi_k}\right)g_k(\psi')\psi' d\psi' \quad (8)$$

where $g_k(\psi)$ is any specified initial distribution of the function P_k. Of particular interest are the initial conditions described by the step function

$$g(\Psi)\begin{cases} = P_{k_j} - P_{k_e} & 0 \leq \Psi \leq \Psi_j \\ = 0 & \Psi_j < \Psi \end{cases} \quad (9)$$

Subject to these conditions, Eq. (8) reduces to

$$P = \frac{P_k - P_{ke}}{P_{kj} - P_{ke}} = \frac{1}{2\xi_k}\exp\left(-\frac{\Psi^2}{4\xi_k}\right)\int_0^{\psi_j} \exp\left(-\frac{\psi'^2}{4\xi_k}\right)I_0 \times \left(\frac{\psi\psi'}{2\xi_k}\right)\psi' d\psi' \quad (10a)$$

and along the axis

$$P_0 = 1 - \exp(-\Psi_j^2/4\xi_k) \quad (10b)$$

The solution (10) is known as the P function, and its values are tabulated in Ref. 15. Physically meaningful solutions can be derived from the general solutions of Eq. (8), provided there exists an inverse transformation from transformed to physical coordinates. Replacing the P_k variables of the previous sections by their physical equivalents, solutions are given in terms of the transformed coordinates

$$\bar{u} = \bar{u}(\xi_v, \Psi) \qquad H = H(\xi_h, \Psi) \qquad Y_i = Y_i(\xi_y, \Psi) \quad (11)$$

Subject to Eq. (7), there exists a relationship between ξ_v, ξ_h, and ξ_y. Substituting for the Ai's and denoting ξ_v by ξ gives

$$\xi_v = \xi(\bar{x}) \qquad d\xi_h/d\xi(x) = Pr^{-1} \qquad d\xi_y/d\xi(x) = Sc^{-1} \quad (12)$$

Equations (11) and (12) combined define a solution in the (ξ, Ψ) plane as

$$\bar{u} = \bar{u}(\xi, \Psi) \qquad H = H(\xi, \Psi) \qquad Y_i = Y_i(\xi, \Psi) \quad (13)$$

The stream function is defined

$$\bar{r}^2 \equiv \int_0^\Psi \frac{\Psi d\Psi}{\bar{\rho}\bar{u}} \quad (14)$$

where $\bar{u} = \bar{u}(\Psi)$ is known from the solution of Eq. (13), and, at any given value of ξ, $\bar{\rho} = \bar{\rho}(\Psi)$ can be obtained from auxiliary equations. The equation of state at constant pressure is

$$\bar{T} = \bar{\rho}\Sigma(Y_i/\bar{w}_i) \tag{15}$$

where the fluid is considered as a mixture of perfect gases. The energy equation is

$$H = \Sigma[Y_i h_i(T)] + u^2/2 \tag{16}$$

Formally, Eq. (16) yields the relation

$$\bar{T} = \bar{T}(H, \bar{u}, Y_i) \tag{17}$$

Then, with $H = H(\Psi)$, $\bar{u} = \bar{u}(\Psi)$, and $Y_i = Y_i(\Psi)$, Eqs. (15) and (17) determine $\bar{\rho} = \bar{\rho}(\Psi)$, and the integrations of Eq. (14) can be performed to produce the desired transformation $\bar{r} = \bar{r}(\Psi)$ for any fixed value of ξ.

The inversion of $\xi \rightarrow \bar{x}$ is based upon Eq. (7), which may be written, in view of Eq. (12), as

$$\bar{x} = \int_0^{\xi} \frac{d\xi}{2(\overline{\rho\epsilon})_{\xi,0}} \tag{18}$$

This transformation can only be carried out where a representative formulation for the coefficient of viscosity $\rho\epsilon$ as a function of the axial coordinate exists or can be derived.

The inverse transformation, as defined by Eq. (14), in conjunction with the equation of state (15) and with (16), introduces the thermodynamic properties of the fluid into the solutions which, in the transformed plane, are obtained independently of such properties. Furthermore, for a fixed value of $\xi = \xi(\bar{x})$, Eqs. (13) and (14) completely determine the radial distribution in the physical plane of all pertinent fluid variables, independently of the physical plane of all pertinent fluid variables, independently of the particular law of variation of the transfer coefficient $\overline{\rho\epsilon}$ chosen. The role of $\rho\epsilon$ is thus restricted to establish where, i.e., at what value of $\bar{x}$, a particular radial distribution takes place. This is clearly a result of taking $\rho\epsilon$ to be a function of the streamwise coordinate only.

Turbulent Flow—Uniform Profiles

For incompressible isothermic flow, where one is concerned with the kinematic viscosity coefficient, a number of formulas are available. Although many objections may be raised regarding the mixing length concept (see, for example, Ref. 16) which underlies both the momentum-transfer theory of Prandtl,[17] and the vorticity transfer theory of Taylor,[18] the success of these theories in comparing favorably with experimental results, in those extrapolated regions where experimental data have not been used for the determination of arbitrary numerical constants, is undeniable.

In compressible flow the eddy viscosity is no longer a measure of the transfer phenomena and must be replaced by a dynamic transfer coefficient, i.e., $\rho\epsilon$, wherein the thermodynamics of the fluid is taking place. As required by the present analysis, $\rho\epsilon$ is taken, at most, as a function of $\bar{x}$ only. In an attempt to derive a semiempirical formula for such a coefficient, the theory must provide for a determination of a numerical value for this coefficient from experimental data. To that effect Eq. (10b) is written in the form

$$\xi_k = -\psi_j^2/4 \ln[1 - P_0(x)] \tag{19}$$

Experimental results provide the axial distribution $P_0 = P_0(x)$; thus, Eq. (19) yields $\xi_k = \xi_k(x)$. From the definition of Eq. (7), the value of $A_k(\overline{\rho\epsilon})$ is established. In particular, when the velocity, total enthalpy, and the mass concentration distributions are given along the axis, Eq. (7) becomes

$$d\xi_v/d\bar{x} = 2(\overline{\rho\epsilon}) \tag{20a}$$

$$d\xi_h/d\bar{x} = 2\overline{\rho\epsilon}_h = 2Pr^{-1}\,\overline{\rho\epsilon} \tag{20b}$$

$$d\xi_y/d\bar{x} = 2\rho\epsilon_y = 2Sc^{-1} \tag{20c}$$

from which $\overline{\rho\epsilon}$, the turbulent Pr, and Sc can be determined.

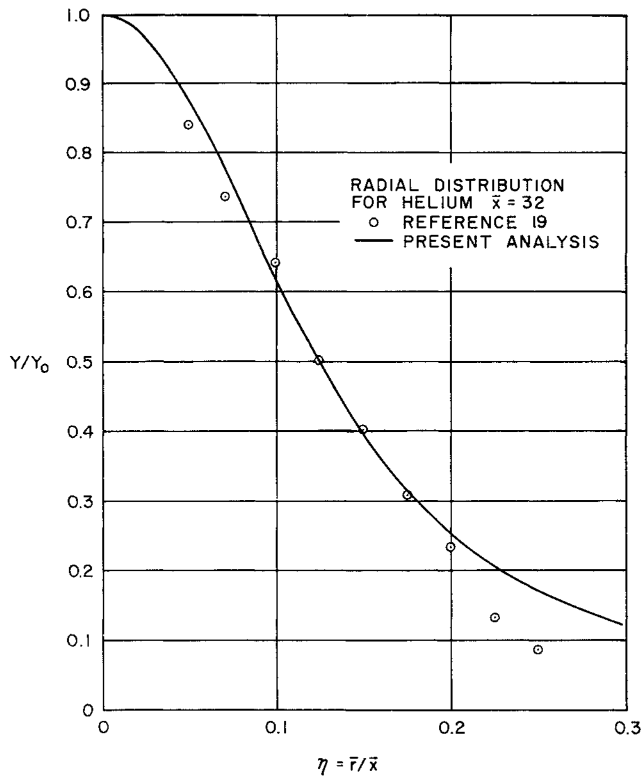

Fig. 2 Radial distribution for helium.

The technique can also be used to demonstrate the validity of the postulate that $\overline{\rho\epsilon} = \overline{\rho\epsilon}(x)$, although the actual form of the function is unknown. Consider an experiment for which the radial distributions, as well as the axial distributions, are known. Equation (19) is then used to determine the value of ξ_k at any point $\bar{x}$ along the axis where $P_0 = P_0(x)$ is known. With the value of ξ_k known, the theoretical radial distribution is computed from Eqs. (10a) and a subsequent inversion by Eq. (5). The application of this technique to the experiments of Keagy and Weller[19] yield the results of Fig. 2. In these experiments, the concentration and velocity of jets of helium, nitrogen, and carbon dioxide issuing vertically from a sharp-edged orifice of 0.128 in. in diameter into still air were determined by a hypodermic tubing of 0.03 in. in diameter, which served both as a sampling probe and as an impact tube. The velocity could not be determined once the impact pressure became low, as in the case of helium; concentration measurements, however, were found to be accurate for all the cases considered. Good agreement is indicated between the computed radial distribution and the experimental results for helium injection into air. Noticeable deviation between analysis and experiment takes place at the tail of the distribution where the analysis predicts higher values than those observed in the experiments. It is clear that for verification of the postulate $\overline{\rho\epsilon} = \overline{\rho\epsilon}(x)$ a large amount of data is required, but for all available experiments that have been considered here, good agreement was shown to exist.

A formulation due to Ferri et al.[12] reduces for the case of $u_e = 0$ to the form

$$\overline{\rho\epsilon} = k_1\bar{b}_{1/2}(\bar{\rho}\bar{u})_0 \tag{21}$$

for the main flow, and

$$\overline{\rho\epsilon} = k_2\bar{x}(\bar{\rho}\bar{u})_0 \tag{22}$$

for the region where the mixing is two-dimensional. The constants k_1 and k_2 must be determined experimentally, and the half-width $b_{1/2}$ is the radius to a point on the jet where the momentum is given by

$$(\bar{\rho}\bar{u}) = \tfrac{1}{2}(\bar{\rho}\bar{u})_0 \tag{23}$$

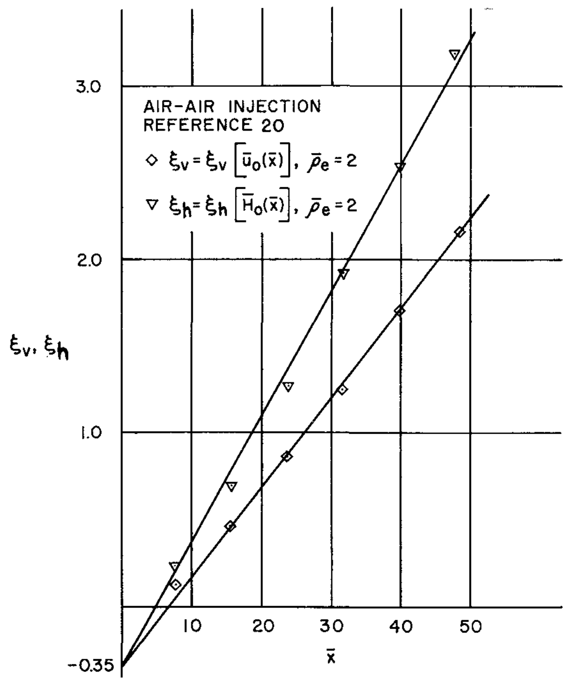

Fig. 3a Linear dependence of ξ_v and ξ_h on $\bar{x}$.

Once these constants are completely defined, solutions for various initial conditions are obtained and compared with available experimental results.

The general problem involves injection of one fluid into another, while the only case considered here is that in which the receiving medium is at rest. This problem is schematically represented in Fig. 1, in which $\bar{u}_e$ is taken to be zero. The solutions for the velocity, stagnation enthalpy, and concentration in terms of the P function notation are

$$\bar{u} = P_0[\Psi_j/(2\xi_v)^{1/2}\ 0]P^*[\Psi_j/(2\xi_v)^{1/2}, \Psi/(2\xi_v)^{1/2}] \quad (24a)$$

$$\frac{H - H_e}{H_j - H_e} = P_0[\Psi_j/(2\xi_h)^{1/2}, 0]P^*[\Psi_j/(2\xi_h)^{1/2}, \Psi/(2\xi_h)^{1/2}] \quad (24b)$$

$$Y = P_0[\psi_j/(2\xi_y)^{1/2}, 0]P^*[\psi_j(2\xi_y)^{1/2}, \psi/(2\xi_y)^{1/2}] \quad (24c)$$

respectively. The $\bar{b}_{1/2}$ in terms of the Ψ coordinates is given by

$$b_{1/2}{}^2 = \int_0^{\Psi_{1/2}} \frac{\Psi d\Psi}{\bar{\rho}\bar{u}} \quad (25)$$

and $\psi_{1/2}$ corresponds to the point in the fluid given by (23). Since k_1 is a constant, its value may be determined from the asymptotic behavior of (21). The asymptotic properties ($\xi \rightarrow \infty$) of the P function, as discussed in Ref. 15, are

$$P^*[\Psi_j/(2\xi)^{1/2}, \Psi/(2\xi)^{1/2}] = \exp\left(-\frac{\Psi^2}{4\xi}\right) \quad (26a)$$

$$P_0[\Psi_j/(2\xi)^{1/2}, 0] = \frac{\Psi_j{}^2}{4\xi} = \frac{1}{2\xi} \quad (26b)$$

where $\psi_j{}^2 = 2$ is obtained from Eq. (14). Since the density asymptotically ($\xi \rightarrow \infty$) becomes uniform, Eq. (23) reads

$$\bar{u}(\xi_v, \Psi_{1/2}) = \tfrac{1}{2}\bar{u}_0(\xi_v)$$

or, from Eq. (24a)

$$P_0P^* = \tfrac{1}{2}P_0$$

Hence,

$$P^*[\Psi_j/(2\xi_v)^{1/2}, \Psi_{1/2}/(2\xi_v)^{1/2}] = 0.5 \quad (27)$$

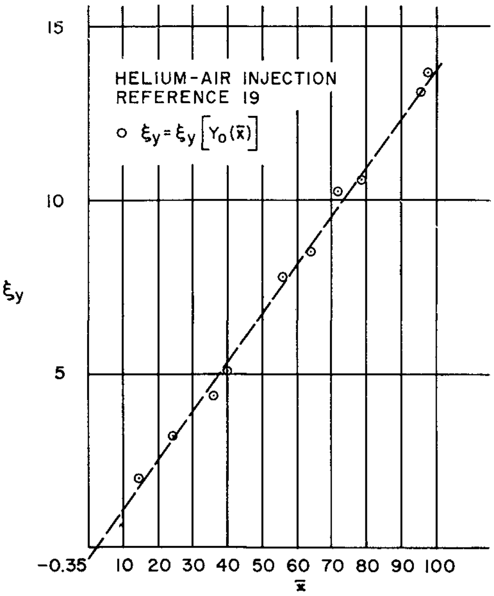

Fig. 3b Linear dependence of ξ_y on $\bar{x}$.

and Eq. (27) can be used as a definition of $\Psi_{1/2}$ for large values of ξ_v. Consider then the asymptotic ($\xi \rightarrow \infty$) behavior of $\bar{b}_{1/2}$, i.e.,

$$\bar{b}_{1/2}{}^2 = \int_0^{\psi_{1/2}} \frac{\Psi d\Psi}{\bar{\rho}\bar{u}} = \frac{1}{\bar{\rho}_e}\int_0^{\psi_{1/2}} \frac{\Psi d\Psi}{P} = \frac{1}{\bar{\rho}_e P_0}\int_0^{\psi_{1/2}} \frac{\Psi d\Psi}{P^*} = \frac{4\xi^2}{\bar{\rho}_e}\left[\exp\left(\frac{\Psi_{1/2}{}^2}{4\xi_v}\right) - 1\right]$$

which, from Eqs. (26a) and (27), becomes

$$\bar{b}_{1/2} = \bar{\rho}_e{}^{-1/2}2\xi_v \quad (28)$$

A substitution of the asymptotic values obtained for $\bar{b}_{1/2}$, $\bar{\rho}_e$, and $\bar{u}_e$ into Eq. (21) gives

$$\overline{\rho\epsilon} = k_1(\bar{\rho}^{-1/2}2\xi_v)(\bar{\rho}_e)\left(\frac{1}{2\xi_v}\right) = k_1\bar{\rho}_e{}^{1/2} \quad (29)$$

An application of Eq. (19) to the experiments of Corrsin and Uberoi[20] indicates that, as can be seen from Fig. 3a, ξ_v and ξ_h are linear in $\bar{x}$ up to a very close region in the neighborhood of the injection point. In Ref. 20 measurements were made of the total head and temperature fields in a round turbulent jet. Heated air was injected vertically into a

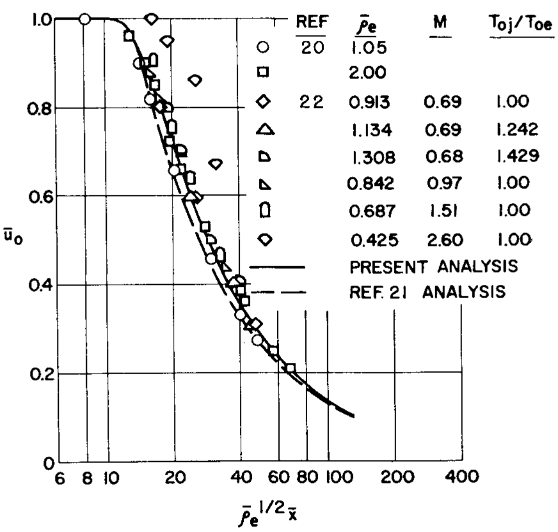

Fig. 4 Axial velocity distribution for turbulent compressible free jets.

quiescent atmosphere with a temperature difference of 27° to 540°R corresponding to density ratios of 1 to 2. It is concluded, therefore, that for compressible flow, as well as for incompressible flow in free jet mixing, the transfer coefficient is constant throughout the flow field, except in the immediate region close to the injection point. Using Eqs. (20) and (29), together with the experimental data of Fig. 3a, the following results are obtained:

$$d\xi_v/d\bar{x} = 0.0518 \qquad \text{or} \qquad \overline{\rho\epsilon} = 0.0259$$

and with $\bar{\rho}_e = 2$,

$$k_1 = 0.0259/(2)^{1/2} = 0.0183$$

From Eq. (20b) and the experimental results

$$d\xi_h/d\bar{x} = 0.0724$$

Hence, assuming k's constants,

$$Pr = d\xi_v/d\xi_h = 0.715 \tag{30}$$

Eq. (29), in terms of the results just obtained, takes the form

$$\overline{\rho\epsilon} = 0.0183\,\bar{\rho}_e^{1/2} \tag{31}$$

Extending Eq. (13) to $x = 0$ permits the integration of Eq. (20), yielding the simple results

$$\xi_v = 0.037\bar{\rho}_e^{1/2}\bar{x} - 0.35 \tag{32}$$

$$\xi_h = 0.051\bar{\rho}_e^{1/2}\bar{x} - 0.35 \tag{33}$$

The application of this method to Keagy and Weller's[19] experiments for helium concentration decay produces the results given in Fig. 3b. Again, the linear dependence of ξ on x for a large region of the flow field is demonstrated. The numerical values obtained are $d\xi_y/d\bar{x} = 0.139$, $(\bar{\rho}_e)^{1/2} = 2.69$, which, from Eq. (20), gives $Sc = 0.708$ and

$$\xi_y = 0.052\bar{\rho}_e^{1/2}\bar{x} - 0.35 \tag{34}$$

The axial decay laws become

$$\bar{u}_0, \frac{H_0 - H_e}{H_j - H_e}, Y_0 = 1 - \exp[-1/(k\bar{x}\bar{\rho}_e^{1/2} - 0.70)] \tag{35}$$

where $k = 0.074$, 0.102, and 0.104, respectively, and $\bar{x}\bar{\rho}_e^{1/2} \geq$ 9.46, 6.86, and 6.73, respectively. The equal sign designates the extent of the initial core length.

A comparison between the axial velocity distribution as given by Eq. (35) and the incompressible solution obtained by Schlichting[21] is shown in Fig. 4. Although the analysis of Ref. 21 follows a different line of investigation, the results compare well, as is indicated in Fig. 4. Results predicted by Eq. (35) are plotted in Figs. 4, 5, and 6, respectively, using $\rho_e^{1/2}\bar{x}$ as the axial coordinate. In these coordinates the axial distributions are independent of the thermodynamics of the fluid, and therefore they may be taken as general curves for the decay of velocity, stagnation enthalpy, and mass concentration in turbulent compressible free jet mixing. Experimental data reported by Keagy and Weller,[19] Corrsin and Uberoi,[20] and Warren[22] are included in these figures. In Ref. 22, air-to-air injection in a horizontal tunnel is studied. Measurements were obtained for heated and unheated jets and for a Mach number range of $M = 0.6$ up to 2.6. The data presented cover a wide range of application. The mass concentration experiments of Ref. 19 include a density ratio of 0.66 in the case of CO_2-air, a ratio of 1.03 in the case of N_2-air, and a ratio of 7.25 in the case of He-air injection. The experiments of Ref. 20 provided data for both stagnation-temperature distribution and for axial-velocity distribution in the low-velocity range. Data are presented for temperature ratio close to unity and for the case of a heated jet, where the temperature of the jet is twice that of the ambient fluid. In both cases air-to-air injection was tested. In the higher-velocity range the experiments of Ref. 22 are included. Data are presented for a Mach number variation from 0.68 to 2.6.

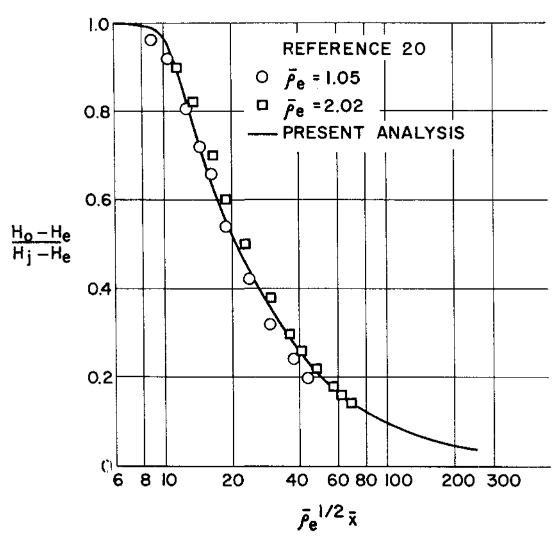

Fig. 5 Axial stagnation enthalpy distribution for turbulent compressible free jets.

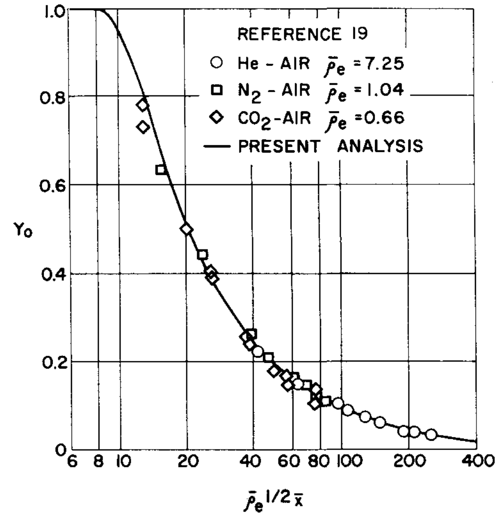

Fig. 6 Axial mass concentration distribution for turbulent compressible free jets.

As is shown in Figs. 4, 5, and 6, except in the case of Mach number equal to 2.6, the large number of experimental data included in these figures compare favorably with the present analysis. Although the dependence of the decay rate on the parameter $\bar{\rho}_e^{1/2}\bar{x}$ is also observed in the $M = 2.6$ case, there is a definite disagreement between the predicted and the measured values of the initial core length. This deviation can be attributed to the existence of pressure disturbances in this region.

Conclusions

The problem of an axially symmetric compressible free jet in a turbulent flow has been analyzed by a linearization of the conservation equations in the von Mises transformation. The resulting momentum, energy, and species conservation equations became uncoupled in the transformed plane, and solutions were obtained for arbitrary initial conditions.

As a consequence of this analysis, the radial distribution of any fluid property at an axial station is found to be a function of the initial conditions and of the value of the property on the axis; it is independent of the particular viscosity law considered. The location of the radial distribution is established once the proper viscosity law $\overline{\rho\epsilon} = \overline{\rho\epsilon}(x)$ is specified.

Use of an extension of Prandtl's formulation of the eddy viscosity to compressible flow indicates that 1) the eddy viscosity for a free compressible jet is independent of the radial coordinate, and 2) the eddy viscosity remains a constant over a large portion of the flow field, excluding the immediate mixing region.

General expressions for the decay of velocity, enthalpy, and concentration are derived, and criteria that determine the extent of the immediate mixing region are also established. The characteristic variable for the streamwise coordinate is shown to be $\rho_e^{1/2}\bar{x}$. In terms of this variable it is shown that the decay of total enthalpy and concentration satisfy the same law very closely and that both decay at a faster rate than the velocity.

The agreement between this analysis and available experimental results is considered good, except in the case of $M = 2.60$. In this case, although the decay does follow the $\rho_e^{1/2}\bar{x}$ dependence, the initial length appears to be longer than the value predicted by the present theory.

References

[1] Kochin, N. E., Kibel, L. A., and Roze', N.V., "Theoretical hydromechanics," Armed Services Technical Information Agency, Translations AD129210, p. 799 (1948).

[2] Kleinstein, G., "An approximation solution for the axisymmetric jet of a laminar compressible fluid," Quart. Appl. Math. **20,** 1, 49–54 (April 1962); also available as Polytechnic Institute of Brooklyn PIBAL Rept. 648, ARL 51 (April 1961).

[3] Oseen, C. W., "On the formula of Stokes" (in German), Arkiv. Math. Aston. Fysik **6** (1910); **7** (1911).

[4] Lewis, J. A. and Carrier, G. F., "Some remarks on the flat plate boundary layer," Quart. Appl. Math. **7,** 228–234 (1949).

[5] Carrier, G. F., "On the integration of equations associated with problems involving convection and diffusion," Tenth International Congress of Theoretical and Applied Mechanics, Stresa, Italy (August 31–September 7, 1960).

[6] von Mises, R., "Bemerkungen zur Hydrodynamik," Z. Angew. Math. Mech. **7,** 425–431 (1927).

[7] Pai, S. I., "Axially symmetric jet mixing of a compressible fluid," Quart. Appl. Math. **10,** 141–148 (1952).

[8] Libby, P. A., "Theoretical analysis of turbulent mixing of reactive gases with application to supersonic combustion of hydrogen," ARS J. **32,** 388–396 (1962).

[9] Prandtl, L., "Bemerkung zur Theories der Freien Turbulenz," Z. Angew. Math. Mech. **22** (1942).

[10] Abramowich, G. N., "Turbulent jets theory," Armed Services Technical Information Agency, Translation AD283858, p. 8 (August 1962).

[11] Kleinstein, G., "On the mixing of laminar and turbulent axially symmetric compressible flows," Polytechnic Institute of Brooklyn PIBAL Rept. 756, ARL 63-108 (June 1963).

[12] Ferri, A., Libby, P. A., and Zakkay, V., "Theoretical and experimental investigation of supersonic combustion," Polytechnic Institute of Brooklyn PIBAL Rept. 713, ARL 62-467 (September 1962).

[13] Carslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids* (Oxford University Press, New York, 1959), p. 260.

[14] Crank, J., *The Mathematics of Diffusion* (Oxford University Press, New York, 1956).

[15] Masters, J. I., "Some applications in physics of the P-function," J. Chem. Phys. **23,** 1865–1874 (1955).

[16] Stewart, R. W., "A new look at the Reynolds stresses," Can. J. Phys. **34** (1956).

[17] Prandtl, L., "Berichte Uber Untersuchungen zur Ausbebildeten Turbulenz," Z. Angew. Math. Mech. **5,** 136 (1925).

[18] Taylor, G. L., "The transport of vorticity and heat through fluid in turbulent motion," Proc. Roy. Soc. **A151,** 494 (1935).

[19] Keagy, W. R. and Weller, A. E., "A study of freely expanding inhomogeneous jets, *Proceedings of the Heat Transfer and Fluid Mechanics Institute* (American Society of Mechanical Engineers, New York, 1949), 89–98.

[20] Corrsin, S. and Uberoi, M. S., "Further experiments on the flow and heat transfer in a heated turbulent air jet," NACA TR 998 (1950).

[21] Schlichting, H., *Boundary Layer Theory* (McGraw-Hill Book Co., Inc., New York, 1955), pp. 483–506.

[22] Warren, W. R., "An analytical and experimental study of compressible free jets," Princeton Univ. Aeronautical Engineering Lab. Rept. 381 (1957).